COSMOLOGICAL SOLUTIONS FOR THE UNIVERSE FILLED WITH MATTER IN VARIOUS STATES AND GAUGE INVARIANCE

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Abstract

We explore at phenomenological level a model of the Universe filled with various kinds of matter characterized by different equations of state. We show that introducing of each kind of matter is equivalent to a certain choice of a gauge condition, the gauge condition describing a medium with a given equation of state. The case of a particular interest is when one kind of matter (de Sitter false vacuum) dominates at the early stage of the Universe evolution while another kind (radiation, or ultrarelativistic gas) dominates at its later stage. We can, therefore, consider different asymptotic regimes for the early and later stages of the Universe existence. These regimes are described by solutions to the Wheeler – DeWitt equation for the Universe with matter in that given state, and, at the same time, in the "extended phase space" approach to quantum geometrodynamics the regimes are described by solutions to a Schrödinger equation associated with a choice of some gauge condition. It is supposed that, from the viewpoint of the observer located at the later stage of the Universe evolution, solutions for a Λ-dominated early Universe would decay.

1. Introduction

It has been realized in recent years that the main problems of the Wheeler – DeWitt quantum geometrodynamics, such as the famous problem of time and a related problem of Hilbert space, cannot be solved without fixing a reference frame. So, in [1] a privileged reference frame was fixed by introducing an incoherent dust which plays the rule of "a standard of space and time".

In [2] the so-called kinematical action was introduced describing a dust reference fluid. Fixing a reference frame results in replacing the Wheeler – DeWitt equation

$$H\Psi = 0 \tag{1.1}$$

by a Schrödinger equation

$$i\frac{\partial\Psi}{\partial t} = H\Psi, \quad \text{or} \quad \mathrm{H}\Psi = \mathrm{E}\Psi. \tag{1.2}$$

(see also the recent paper [3], where the authors start from Schrödinger, or evolutionary quantum gravity; for a review on the problem of time and gauge invariance in quantum cosmology see [4, 5]).

The same purpose (the replacement of the Wheeler – DeWitt static picture of the world by a Schrödinger evolutionary dynamics) was achieved in an explicitly gauge noninvariant approach named "quantum geometrodynamics in extended phase space" [6, 7, 8, 9]. In these works it was demonstrated that, firstly, quantum theory of the Universe with non-trivial topology cannot be constructed in a gauge invariant way, and, secondly, the introduction of a gauge fixing term into the gravitational action can be interpreted as the existence of some media with a certain equation of state in the Universe.

On the other side, even in the limits of canonical quantization of gravity the Wheeler – DeWitt equation can be reduces to a Schrödinger-like equation $\tilde{H}\Psi = E\Psi$, where $\tilde{H} - E = 0$ is a new form of the Hamiltonian constraint, and E is a conserved quantity, which appears from phenomenological consideration of matter fields with various equations of state, including de Sitter false vacuum. Solving this equation, one would find an energy spectrum for this sort of matter. This approach was discussed, in particular, in the works by Dymnikova and Fil'chenkov [10, 11]. The purpose of this talk is to compare the both approaches and show that, in some sense, introducing of each kind of matter is equivalent to a certain choice of a gauge condition.

2. Canonical quantization approach

To give a qualitative discussion of the question, I shall advisedly restrict myself to the simplest isotropic closed model with the action

$$S = -\int dt\,\left(\frac{1}{2}\frac{a\dot{a}^2}{N} - \frac{1}{2}Na + Na^3\Lambda\right) + S_{(mat)}, \quad S_{(mat)} = -\int dt\,Na^3\varepsilon(a). \tag{2.1}$$

Matter fields are presented in this model phenomenologically, without a clear indication on the nature of the fields, by its energy density $\varepsilon(a)$, the dependence of ε on a determining its equation of state. It is easy to check that for

$$\varepsilon(a) = \frac{\varepsilon_n}{a^n} \tag{2.2}$$

$\varepsilon_n = \text{const}$ for each kind of matter, one obtains the equation of state

$$p = \left(\frac{n}{3} - 1\right)\varepsilon. \tag{2.3}$$

Thus, this phenomenological consideration allows one to include most known kinds of matter characterized by their equations of state for various n [10]. For radiation, or ultrarelativistic gas we would have

$$\varepsilon(a) = \frac{\varepsilon_4}{a^4}, \quad p = \frac{\varepsilon}{3}. \tag{2.4}$$

We also introduce explicitly the cosmological constant Λ, which is associated with de Sitter vacuum with the equation of state $p = -\varepsilon$. The Hamiltonian constraint for this model reads

$$\frac{1}{2a}p_a^2 + \frac{1}{2}a - a^3\Lambda - a^3\varepsilon(a) = 0. \tag{2.5}$$

To obtain equivalent forms of the gravitational constraint, one could consider an arbitrary parameterization of the gauge variable,

$$N = v(\tilde{N},\, a), \tag{2.6}$$

and the most general form of the constraint is

$$\frac{\partial v}{\partial \tilde{N}}\left(\frac{1}{2a}p_a^2 + \frac{1}{2}a - a^3\Lambda - a^3\varepsilon(a)\right) = 0. \tag{2.7}$$

Choosing $v(\tilde{N},\, a) = \tilde{N}a$, one would get the equation

$$\frac{1}{2}p_a^2 + \frac{1}{2}a^2 - a^4\Lambda - a^4\varepsilon(a) = 0. \tag{2.8}$$

For a universe filled with radiation in which $\Lambda = 0$, it gives a Schrödinger-like equation

$$-\frac{1}{2}\frac{d^2\Psi}{da^2} + \frac{1}{2}a^2\Psi = E\Psi, \quad E = \varepsilon_4. \tag{2.9}$$

On the other hand, one can consider a universe with a non-zero cosmological constant, but without matter. Choosing another parameterization, $v(\tilde{N},\, a) = \dfrac{\tilde{N}}{a^3}$, one would come to the equation

$$\frac{1}{2a^4}p_a^2 + \frac{1}{2a^2} - \Lambda - \varepsilon(a) = 0. \tag{2.10}$$

Writing down a quantum version of the constraint (2.10), one faces the ordering problem. The ordering should be chosen for the Hamiltonian operator to be Hermitian,

$$-\frac{1}{2}\frac{1}{a^2}\frac{d}{da}\left(\frac{1}{a^2}\frac{d\Psi}{da}\right) + \frac{1}{2a^2}\Psi = \Lambda\Psi, \tag{2.11}$$

or,

$$-\frac{1}{2}\frac{1}{a^4}\frac{d^2\Psi}{da^2}+\frac{1}{a^5}\frac{d\Psi}{da}+\frac{1}{2a^2}\Psi=\Lambda\Psi. \tag{2.12}$$

Though the constraints (2.5), (2.8) and (2.10) are completely equivalent at the classical level, the equations (2.9) and (2.12) are not. Their solutions belong to different Hilbert spaces, so that even measures in inner products are different. In the case of Eq. (2.9) the measure is trivial, $M(a)=1$, while in the second case, for the Hamiltonian to be Hermitian one should choose $M(a)=a^2$. It is not clear, if there is any relation between solutions of the equations (2.9) and (2.12). Similarly, we could write down equations for other sorts of matter. Indeed, for any parametrization

$$v(\tilde{N},\,a)=\tilde{N}a^{n-3} \tag{2.13}$$

we would obtain a constraint

$$\frac{1}{2}a^{n-4}p_a^2+\frac{1}{2}a^{n-2}-\varepsilon_n=0, \tag{2.14}$$

and an appropriate Schrödinger equation

$$-\frac{1}{2}a^{\frac{n}{2}-2}\frac{d}{da}\left(a^{\frac{n}{2}-2}\frac{d\Psi}{da}\right)+\frac{1}{2}a^{n-2}\Psi=E\Psi,\quad E=\varepsilon_n, \tag{2.15}$$

($n=0$ for de Sitter false vacuum, $n=4$ for radiation-dominated universe, etc.). E stands for energy eigenvalues of the given kind of matter. Let me emphasize that all these equations can be obtained in the limits of the Dirac – Wheeler – DeWitt canonical quantization scheme.

3. Extended phase space approach

On the contrast, in the extended phase space approach we can work with a gauged action for pure gravitation, without matter and Λ-term

$$S_{(gauged)}=-\int dt\,\left[\frac{1}{2}\frac{a\dot{a}^2}{N}-\frac{1}{2}Na+\pi_0\left(\dot{N}-\frac{df}{da}\dot{a}\right)+N\dot{\bar{\theta}}\bar{\theta}\right], \tag{3.1}$$

or, for an arbitrary parameterization of the gauge variable $N=v(\tilde{N},\,a)$,

$$S_{(gauged)}=-\int dt\,\left[\frac{1}{2}\frac{a\dot{a}^2}{v(\tilde{N},\,a)}-\frac{1}{2}v(\tilde{N},\,a)a+\pi_0\left(\dot{\tilde{N}}-\frac{df}{da}\dot{a}\right)+w(\tilde{N},\,a)\dot{\bar{\theta}}\bar{\theta}\right], \tag{3.2}$$

where a differential form of a gauge condition $\tilde{N}-f(a)=0$ is used, $w(\tilde{N},\,a)\equiv v(\tilde{N},\,a)\left(\frac{\partial v}{\partial\tilde{N}}\right)^{-1}$, and θ, $\bar{\theta}$ are the Faddeev – Popov ghosts, though a ghost sector will not affect further consideration.

In general, a gauged action gives the gauged Einstein equations

$$R^{\nu}_{\mu}-\frac{1}{2}\delta^{\nu}_{\mu}R=\kappa\left(T^{\nu}_{\mu(mat)}+T^{\nu}_{\mu(obs)}+T^{\nu}_{\mu(ghost)}\right), \tag{3.3}$$

which involve, apart from energy-momentum tensor of matter fields $T^{\nu}_{\mu(mat)}$, additional terms $T^{\nu}_{\mu(obs)}$ and $T^{\nu}_{\mu(ghost)}$ obtained by varying the gauge-fixing and ghost action, respectively. $T^{\nu}_{\mu(obs)}$ and $T^{\nu}_{\mu(ghost)}$ are not true tensors, but quasi-tensors depending on a chosen gauge conditions.

A consistent quantization procedure implies derivation of a Schrödinger equation from a path integral with the gauged action without asymptotic boundary conditions, the latter ones are thought to be irrelevant for a closed universe [6, 7]. The Schrödinger equation for a physical part of the wave function reads

$$\left.\left[-\frac{1}{2}\sqrt{\frac{v(\tilde{N},\,a)}{a}}\frac{d}{da}\left(\sqrt{\frac{v(\tilde{N},\,a)}{a}}\frac{d\Psi}{da}\right)+\frac{1}{2}v(\tilde{N},\,a)a\Psi\right]\right|_{\tilde{N}=f(a)}=E\Psi. \tag{3.4}$$

The operator in the left-hand side of (3.4) is an analogue of Laplacian, however, as we can see, it is gauge-dependent,

$$E=-\int\sqrt{-g}\,T^{0}_{0(obs)}\,d^3x. \tag{3.5}$$

For our model

$$T^{\nu}_{\mu(obs)}=\mathrm{diag}\left(\varepsilon_{(obs)},\,-p_{(obs)},\,-p_{(obs)},\,-p_{(obs)}\right); \tag{3.6}$$

$$\varepsilon_{(obs)}=\left.\frac{\dot{\pi}_0}{2\pi^2a^3}\left(\frac{\partial v}{\partial\tilde{N}}\right)^{-1}\right|_{\tilde{N}=f(a)}; \tag{3.7}$$

$$p_{(obs)}=\varepsilon_{(obs)}\left.\frac{a}{3v(\tilde{N},\,a)}\left(\frac{\partial v}{\partial a}+\frac{\partial v}{\partial\tilde{N}}\frac{df}{da}\right)\right|_{\tilde{N}=f(a)}. \tag{3.8}$$

The last formula gives the equation of state for a medium describing by the quasi-tensor $T^{\nu}_{\mu(obs)}$ and depending on a chosen parameterization and gauge, meanwhile Eq. (3.4) determining the energy spectrum of this medium. For the parametrization (2.13) and the gauge condition $\tilde{N}=1$ we immediately obtain the equation of state (2.3) from (3.8) and Eq. (2.15) from (3.4). Choosing in (3.4) $v(\tilde{N},\,a)=\tilde{N}a$, we would come again to Eq. (2.9), while the choice $v(\tilde{N},\,a)=\dfrac{\tilde{N}}{a^3}$ gives Eq. (2.12). In other words, Eq. (2.9), which is believed to describe a universe filled with radiation, corresponds to the conformal time gauge $N=a$, and Eq. (2.12), describing a universe with non-zero cosmological constant, corresponds to the gauge condition $N=\dfrac{1}{a^3}$. The results are completely equivalent to those of Section 2.

Solutions to (2.9), (2.12) can be considered as those of the Wheeler – DeWitt equation for the Universe with matter in a given state, and, at the same time, in the extended phase space approach as solutions of a Schrödinger equation associated with some gauge condition. At the level of the equations and their solutions there is no distinctions between the two approaches presented in Sections 2 and 3.

We can see that the medium describing by the quasi-tensor $T^{\nu}_{\mu(obs)}$ simulates the properties of a substance characterized by a given equation of states, in the sense that the obtained equations are the same. Therefore, consideration of each kind of matter is equivalent, on a phenomenological level, to a certain choice of a gauge condition. It shows that the canonical quantization scheme, outlined in Section 2 for arbitrary parametrization, cannot be considered as a gauge-invariant scheme. The condition for a gauge variable $\tilde{N} = 1$ is used in this scheme implicitly.

4. Solutions for the Universe filled with two-component medium

In the extended phase space approach we can explore even more complicated and exotic cases. If in the limits of the approach considered in Section 2 we can seek for eigenvalues spectrum for a only one kind of matter in the Universe, and the presence of other matter fields can be taken into account through additional terms in an effective potential, the extended phase space approach allows us to study spectra of multicomponent media. Let us turn to the parameterization

$$N = v(\tilde{N},\, a) = \tilde{N}\left(a + \frac{1}{a^3}\right). \tag{4.1}$$

At small values of a we have $v(\tilde{N},\, a) = \dfrac{\tilde{N}}{a^3}$, while at large a $v(\tilde{N},\, a) = \tilde{N}a$. The equation of state (3.8) under the condition $\tilde{N} = 1$ gives

$$p_{(obs)} = \varepsilon_{(obs)}\frac{a^4 - 3}{3(a^4 + 1)}. \tag{4.2}$$

Again, in the limit $a \to 0$ we get $p_{(obs)} = -\varepsilon_{(obs)}$, the equation of state for de Sitter false vacuum, and in the limit $a \to \infty$ the equation of state is that of radiation, $p_{(obs)} = -\dfrac{\varepsilon_{(obs)}}{3}$. The Schrödinger equation (3.4) for this case reads

$$-\frac{1}{2}\left(1 + \frac{1}{a^4}\right)\frac{d^2\Psi}{da^2} + \frac{1}{a^5}\frac{d\Psi}{da} + \frac{1}{2}\left(a^2 + \frac{1}{a^2}\right)\Psi = E\Psi. \tag{4.3}$$

The equations (2.12) and (2.9) can be obtained as asymptotic limits of (4.3) at $a \to 0$ and $a \to \infty$, respectively. Eq. (4.3) describes a universe filled with a two-component medium, de Sitter vacuum dominating at the early stage of the Universe evolution, and radiation dominating at its later stage.

To understand the behaviour of solutions to (4.3) as well as solutions for limiting cases (2.12) and (2.9) one can seek for numerical solutions of these equations. A well-known method of finding approximate eigenvalues and eigenfunctions of a Hermitian operator consists in expanding unknown functions onto a given basis in appropriate Hilbert space. As was already mentioned, solutions to the equations (2.12), (2.9) and (4.3) belong to different Hilbert spaces. Say, the solutions to Eq. (2.12) exist in the Hilbert space $\mathcal{H}_1$ with the measure $M(a) = a^2$, while the solutions to Eq. (2.9) "live" in the Hilbert space $\mathcal{H}_2$ with the measure $M(a) = 1$. We should remember that if we try to expand solutions to Eq. (2.12) onto a basis in $\mathcal{H}_2$, we would obtain, in general, complex eigenvalues, that tells about non-stability of the solutions corresponding to a vacuum-dominated early universe and their tendency to decay from the viewpoint of the observer located at the later stage of the Universe evolution and using conformal time gauge $N = a$.

We expect that in a vacuum-dominated universe (in the limit $a \to 0$) a peak of probability distribution for solutions to Eq. (2.12) for larger (by its absolute values) eigenvalues tends to shift to small values of a. Therefore, for larger eigenvalues of the cosmological constant Λ, which we find solving Eq. (2.12), the Universe appears to be localized in the region of small a. On the contrary, in a universe filled with radiation ($a \to \infty$), which is described by solutions to Eq. (2.9), a peak of probability distribution for larger eigenvalues tends to shift to large values of the scale factor. So, when the energy of this kind of matter increases, there may be enough probability for the scale factor to reach large values. Detailed analysis of results of numerical calculations will be published elsewhere.

5. Concluding remarks

Now we return to the main question of this talk. We have demonstrated that an equation, describing the Universe filled with a certain sort of matter, can be obtain in the framework of the extended phase space approach under a suitable choice of a gauge condition. On the other hand, to investigate an energy spectrum for this sort of matter in the Wheeler – DeWitt quantum geometrodynamics, one has to choose a certain form of the gravitational constraint, i.e. a certain parameterization of a gauge variable. Moreover, the Dirac – Wheeler – DeWitt quantization procedure implies implicitly a condition on a gauge variable $\tilde{N} = 1$. As has been shown in [6, 9, 12], the choice of parameterization and that of a gauge condition have a unified interpretation: together they fix a reference frame in which we study geometry of the Universe and matter distribution in it.

We come to the picture which is very different from "objective physics" we would like to deal with. To explore some kind of matter in a quantum universe, one has to tune up a measuring instrument in a certain way, choosing parameterization and gauge. This changes the form of our equations. Should we consider this fact as an indication that our theory is wrong or accept this fact like we accept an uncertainty principle? Let me finish on this point.

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